

A not-so-normal mode decomposition

Michael M. Wolf

Max-Planck-Institute for Quantum Optics, Hans-Kopfermann-Str. 1, D-85748 Garching, Germany.

(Dated: March 10, 2008)

We provide a generalization of the normal mode decomposition for non-symmetric or locality constrained situations. This allows for instance to locally decouple a bipartitioned collection of arbitrarily correlated oscillators up to elementary pairs into which all correlations are condensed. Similarly, it enables us to decouple the interaction parts of multi-mode channels into single-mode and pair-interactions where the latter are shown to be a clear signature of squeezing between system and environment. In mathematical terms the result is a canonical matrix form with respect to real symplectic equivalence transformations.

The normal mode decomposition is a ubiquitous and indispensable tool in physics and engineering. It allows us to transform into frames where seemingly complex systems decouple into elementary units each of which can be tackled individually. Consider for instance a collection of harmonically coupled oscillators, classical or quantum. Then the normal mode decomposition provides us with a canonical transformation which decouples the state of all oscillators into independent normal modes. Similarly, by the seminal work of Williamson [1], the evolution of a system governed by a quadratic Hamiltonian can be decoupled into independent elementary parts. In both cases—states and evolutions—this decoupling requires a global transformation. In many situations, however, additional restrictions prevent us from simplifying the structure of a system by such a transformation. We might for instance want to preserve the amount of correlations or entanglement of a bipartitioned collection of oscillators. Or, in the case of evolution, we might be faced with dissipation, i.e., an inaccessible environment.

In this work we provide a generalization of the normal mode decomposition which is applicable in these and other situations and allows again to decouple correlations and interactions into elementary parts. This will reveal a remarkable structure of both, correlations and interactions, in which elementary units turn out to be pairs of modes rather than single modes. Hence, as for the normal mode decomposition, we get again a significant simplification in many contexts dealing with many-body harmonic or bosonic systems, albeit with a richer structure.

In the first part we will state and prove the main result in terms of matrix analysis where it amounts to a normal form with respect to symplectic equivalence transformations, i.e., a symplectic analogue of the singular value decomposition—inspired by recent advances in symplectic geometry [2, 5]. In the second part we will then apply it to the above mentioned cases of bipartite correlations and dissipative evolutions (depicted in Figs. 1,2) and connect it to known results.

PRELIMINARIES

Before introducing some basic notions let us mention that although we will have quantum systems in mind in the following, all results hold for classical systems in exactly the

same way. Similarly, note that the oscillators do not have to be mechanical but might as well correspond to electromagnetic field modes, charge-phase oscillations in Josephson junctions or collective spin fluctuations, e.g., in atomic ensembles. In each of these cases we have pairs of canonically conjugate variables obeying the same (commutation) relation as position and momentum.

Consider now n quantum mechanical oscillators characterized by a set of momentum and position operators $(P_1, \dots, P_n, Q_1, \dots, Q_n) =: R$ which obey the canonical commutation relations $[R_k, R_l] = i\sigma_{kl}$, with

$$\sigma = \begin{pmatrix} 0 & -\mathbb{1}_n \\ \mathbb{1}_n & 0 \end{pmatrix} \quad (1)$$

the symplectic matrix. A canonical/symplectic transformation maps $R_k \mapsto \sum_l S_{kl} R_l$ such that the commutation relations (or classical Poisson brackets) are preserved, i.e., $S\sigma S^T = \sigma$. We will denote the group of real symplectic transformation on n modes by $Sp(2n)$. For a basic introduction into symplectic transformations and the appearance and use of canonically conjugate variables in quantum information theory we refer the reader to [3] and [4].

The essence of the ordinary normal mode decomposition [1] is the fact that for any positive definite matrix $X \in \mathbb{R}^{2n \times 2n}$ there is an $S \in Sp(2n)$ such that

$$SXS^T = \text{diag}(\nu_1, \dots, \nu_n, \nu_1, \dots, \nu_n). \quad (2)$$

In case X represents a Hamiltonian $H = \sum_{kl} X_{kl} R_k R_l$ the ν_k are the normal mode frequencies. If $X_{kl} = \langle \{R_k - \langle R_k \rangle, R_l - \langle R_l \rangle\}_+ \rangle$ is a covariance matrix, then $(\nu_k - 1)/2$ is the mean occupation number (phonons/photons) in the k 'th normal mode.

CANONICAL FORM

We aim at deriving a canonical form for general (not necessarily symmetric) matrices $X \in \mathbb{R}^{2n \times 2n}$ under symplectic equivalence transformations $X \mapsto S_1 X S_2$. To this end we will first construct a set of invariants which play a role similar to the normal mode frequencies ν_k :

Proposition 1 (Invariants) *The eigenvalues of $\Sigma(X) := X\sigma X^T\sigma^T$ are invariant with respect to symplectic equivalence transformations of the form $X \mapsto S_1 X S_2$.*

The proof of this statement is simple. We have just to exploit that $S_i\sigma S_i^T = \sigma$ and that for any two matrices AB and BA have the same non-zero spectrum.

Note that the entire spectrum of $\Sigma(X)$ is two-fold degenerate, i.e., $\text{spec}(\Sigma) = \{\lambda_1, \dots, \lambda_n, \lambda_1, \dots, \lambda_n\}$ [2] and in addition complex eigenvalues come in conjugate pairs $\lambda, \bar{\lambda}$ (as it holds for every real matrix). If X is positive definite we recover, in fact, the normal mode frequencies $\nu_k = \sqrt{\lambda_k}$ so that all invariants are positive real numbers in this case. The occurrence of complex eigenfrequencies is, in fact, a well known phenomenon in many fields of physics in particular where harmonic approximations are used in intermediate energy regimes. Examples can be found in contexts from molecular condensates [6] to gravitational waves [7] and sonic black holes [8].

While Prop.1 evidently holds for arbitrary rectangular and possibly singular matrices X , we will for the sake of simplicity restrict ourselves to non-singular square matrices in the following. Our aim is to show that $\lambda_1, \dots, \lambda_n$ are the only invariants and that they essentially determine the normal-form of X with respect to symplectic equivalence transformations:

Proposition 2 (Canonical form) *For every nonsingular matrix $X \in \mathbb{R}^{2n \times 2n}$ there exist real symplectic transformations S_1, S_2 such that*

$$S_1 X S_2 = \begin{pmatrix} \mathbb{1}_n & 0 \\ 0 & J \end{pmatrix}, \quad J = \begin{pmatrix} J_1(\lambda_1) & 0 & \dots \\ 0 & J_2(\lambda_2) & \dots \\ \vdots & \vdots & \ddots \end{pmatrix}, \quad (3)$$

where each J_k is a real Jordan block [9] corresponding to either a complex conjugate pair $\{\lambda, \bar{\lambda}\}$ of eigenvalues of $\Sigma(X)$ or to one of its real eigenvalues. In the former case the diagonal of $J_k(\lambda)$ is build up out of real 2×2 blocks of the form

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}, \quad \lambda = a + ib. \quad (4)$$

Proof. First note that $\Sigma(X)$ is a skew-Hamiltonian matrix, i.e., $(\Sigma\sigma)^T = -(\Sigma\sigma)$ (we drop the dependence on X in the following). For every real skew-Hamiltonian matrix there exists a real symplectic similarity transformation such that $S\Sigma S^{-1} = -(M \oplus M^T)$ is block diagonal [2]. Exploiting in addition that every real matrix M can be written as a product of two real symmetric matrices $AB = M$ [10] we can write

$$(SX)\sigma(SX)^T\sigma = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}^2 =: W^2. \quad (5)$$

Now define a transformation S' by imposing $SX S' = W$. In fact, S' is symplectic as can be seen from

$$S'^T \sigma S' = W^T \sigma^T [\sigma(SX)^{-T} \sigma(SX)^{-1}] W = \sigma, \quad (6)$$

where we have used that $\sigma^T \sigma = \mathbb{1}$, $W^T \sigma^T = \sigma W$ and that the expression in squared brackets is by Eq.(5) equal to W^{-2} . $W^T \sigma^T = \sigma W$ is easily seen by exploiting the A, B -block structure of W . Note that Eq.(6) is the point in proof where we use non-singularity of X .

To proceed we exploit the subgroup $GL(n) \subset Sp(2n)$, i.e., the fact that for every real invertible matrix G , the block matrix $G^{-1} \oplus G^T$ is symplectic [3]. Multiplying $SX S'$ from the left with $A^{-1} \oplus A^T$ and from the right with σ , which is a symplectic transformation in its own right, we obtain $\mathbb{1}_n \oplus (-A^T B)$. This can be brought to the claimed form in Eq.(3) via a symplectic similarity transformation by $G^{-T} \oplus G$. Here we use the real Jordan canonical form $J = G(-A^T B)G^{-1}$ in which complex conjugate pairs of eigenvalues correspond to real 2×2 matrices of the form in Eq.(4) (cf.[11]). It remains to show that the spectrum of $\Sigma(X)$ is a doubling of the spectrum of J . By Prop.1 we have that $\text{spec}(\Sigma(X)) = \text{spec}(\Sigma(\mathbb{1} \oplus J))$ so that the identity $\Sigma(\mathbb{1} \oplus J) = J^T \oplus J$ completes the proof. \square

Some remarks on the normal form in Eq.(3) are in order. First note that it is minimal in the sense that the number of continuous parameters cannot be further reduced by symplectic equivalence transformations as they are all invariants due to Prop.1. Similarly looking at the invariants tells us that an entirely diagonal normal form, analogous to the usual normal mode decomposition, cannot exist in general as real diagonal matrices have only real invariants. Hence, there is no way of diagonalizing the remaining 2×2 blocks since they correspond to complex λ 's. Concerning the possible appearance of defective parts in the Jordan blocks [9] we note that, as usual, they are not stable with respect to perturbations. The fact that matrices with non-defective normal form are dense can be seen by noting that for any $X \in \mathbb{R}^{2n \times 2n}$ there exists an X_ϵ arbitrary close to it such that $\text{spec}(\Sigma(X_\epsilon))$ is only two-fold degenerate (and non-singular). The corresponding J has no degeneracy and is thus non-defective.

The presented canonical form can be regarded as a generalization of the seminal results by Williamson [1] and its extensions [12] on normal forms of symmetric (not necessarily positive) matrices under symplectic transformations. For positive definite matrices X the canonical form in Eq.(3) and the usual normal mode decomposition in Eq.(2) coincide up to a simple squeezing transformation and the invariants are related via $\lambda_k = \nu_k^2$.

In the following we will discuss two applications of the above result in contexts where the two sets of modes on which S_1 and S_2 act on either correspond to two different parties (Alice and Bob, say) or to input and output of a quantum channel. Note that in both cases the two sets need not be of the same physical type. A prominent example of that form is a set of light modes coupled to modes of collective spin fluctuations in atomic ensembles [13].

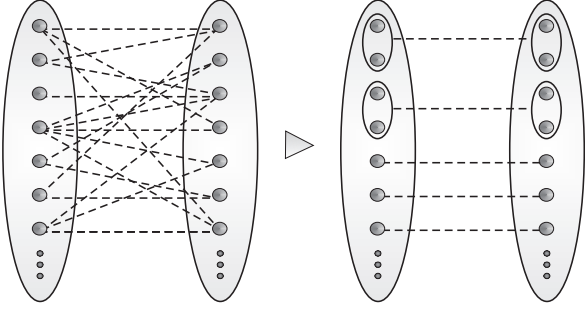


FIG. 1: *Left*: let Alice and Bob each have a collection of oscillators with arbitrary correlations between them (indicated by dashed lines). *Right*: there exist local canonical transformations which condense the correlations into elementary units (single modes or pairs) and eliminate all cross-correlations.

DECOUPLING AND CONDENSING CORRELATIONS

Consider a bipartitioned collection of $n + n$ modes as in Fig.1. The covariance matrix $\Gamma \in \mathbb{R}^{4n \times 4n}$ can then be partitioned into blocks

$$\Gamma = \begin{pmatrix} \Gamma_A & X \\ X^T & \Gamma_B \end{pmatrix}, \quad (7)$$

where Γ_A , Γ_B are the local covariance matrices and X describes correlations between the two parts. A local symplectic transformation $\Gamma \mapsto (S_A \oplus S_B)\Gamma(S_A \oplus S_B)^T$ transforms the correlation block as $X \mapsto S_A X S_B^T$. Hence, Prop.2 can be directly applied to decouple and condense the correlations. In the generic case of non-singular, non-defective X we are then left with correlated single modes whose correlations are characterized by real invariants λ and with correlated pairs corresponding to complex λ with a correlation block of the form in Eq.(4). All the cross-covariances between opposite subsets which do not correspond vanish (see Fig.1).

A particular known instance of this result is the case of pure Gaussian quantum states. For pure bipartite states correlations, which are then due to entanglement, and local spectral properties determine each other. This is the content of the Schmidt decomposition which in terms of the covariance matrix implies that Γ_A , Γ_B and X can then be simultaneously diagonalized by local symplectic transformations such that $\nu_k = \sqrt{1 - \lambda_k}$ ($\lambda_k \leq 0$) is the mean particle number in the k 'th normal mode of each site [14, 19].

This pure state normal form simplified investigations in various directions like security proofs in quantum cryptography [15] or the transformation [16], localization [17] and characterization [18] of entanglement. Prop.2. now provides the analogous normal form for mixed states, which again can be regarded as a condensation of correlations. As such, it might be a useful first step in quantum information protocols which use correlations as a resource.

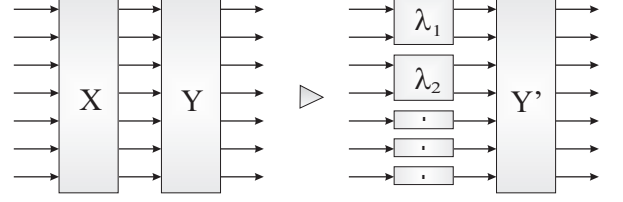


FIG. 2: *Left*: A multi-mode Gaussian channel consists out of an interaction part X , which directly couples the input modes, and an addition of input-independent noise Y . *Right*: The interaction part is decoupled into single-mode and two-mode parts by applying canonical transformations before and after the channel. Two-mode interactions corresponding to a complex invariant λ only occur if the global system-plus-environment evolution involves squeezing.

DECOUPLING OF INTERACTIONS

Let us now consider dissipative evolutions of multi-mode quantum systems. An important class of such evolutions are those where system plus environment undergo a global canonical transformation. For the system (with traced out environment) this leads to so-called Gaussian or quasi-free channels [19, 20] realized by optical fibres or, if the transformation is in time rather than in space, by quantum memories build upon atomic ensembles [21].

Gaussian channels are characterized by a pair of matrices $X, Y \in \mathbb{R}^{2n \times 2n}$ satisfying the constraint $iX^T \sigma X + Y \geq i\sigma$. The covariance matrix evolves then according to

$$\Gamma \mapsto X^T \Gamma X + Y. \quad (8)$$

That is, X can be regarded as characterizing direct interactions between the modes and Y is a noise-term which is input-independent. If we now apply a symplectic transformation before and after the channel then $X, Y \mapsto S_1 X S_2, S_2^T Y S_2$. We can thus again exploit Prop.2 in order to simplify the structure of the interaction part of the evolution. This way of encoding and decoding information sent through the channel has been successfully exploited in the context of channel capacities of single-mode channels [22] for which it leads to a simple normal form [23]. For multi-mode channels it allows us, in the generic case, to reduce X to two-mode interactions of the form in Eq.(4) and single-mode parts (see Fig.2).

The appearance of pair interactions corresponding to complex invariants λ is, in fact, a signature of a non-number preserving system-environment interaction. In order to see this recall that a global number-preserving transformation can be written as

$$S = \begin{pmatrix} C & D \\ -D & C \end{pmatrix}, \quad (9)$$

where $C + iD$ is a unitary and the block structure in Eq.(9) refers to a decomposition of phase space into position and momentum space (rather than system and environment). For the reduced system evolution this leads to an X which has the

same structure as S but without the restriction of the matrices being real and imaginary parts of a unitary. Let us denote the corresponding blocks in X by c and d and calculate

$$\Sigma(X) = \begin{pmatrix} dd^T + cc^T & dc^T - cd^T \\ cd^T - dc^T & dd^T + cc^T \end{pmatrix}. \quad (10)$$

As $\Sigma(X)$ is Hermitian it has indeed only real eigenvalues λ_k which shows that pair interactions in the normal form which correspond to complex λ 's witness a squeezing-type interaction between system and environment.

Acknowledgments

The author thanks A.S. Holevo for many inspiring discussions on the topic.

-
- [1] J. Williamson, Am. J. Math **58**, 141 (1936); **59**, 599 (1937); **61**, 897 (1939).
 - [2] W.C. Waterhouse, Linear Algebra Appl. **396**, 385 (2005); H. Fassbender, D.S. Mackey, N. Mackey, H. Xu, Linear Algebra Appl. **287**, 125 (1999).
 - [3] Arvind, B. Dutta, N. Mukunda, R. Simon, Pramana **45**, 471 (1995); arXiv:quant-ph/9509002.
 - [4] S.L. Braunstein, P. van Loock, Rev. Mod. Phys. **77**, 513 (2005).
 - [5] Kh. D. Ikramov, E.A. Zaikova, Univ. Ser. XV Vychisl. Mat. Kibernet. **1**, 23 (1988).
 - [6] H.Y. Ling, H. Pu, B. Seaman, Phys. Rev. Lett. **93**, 250403 (2004).
 - [7] K.D. Kokkotas, B.G. Schmidt, Living Rev. Rel. **2** 2 (1999).
 - [8] L.J. Garay, J.R. Anglin, J.I. Cirac, P. Zoller, Phys. Rev. Lett. **85**, 4643 (2000).

- [9] Each real Jordan block has the form

$$J(\lambda) = \begin{pmatrix} \Lambda & \mathbb{1}_2 & & \\ & \Lambda & \ddots & \\ & & \ddots & \mathbb{1}_2 \\ & & & \Lambda \end{pmatrix}, \quad \Lambda = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}, \quad \lambda = a + ib,$$

and it is called non-defective if $J(\lambda) = \Lambda$.

- [10] G. Frobenius, Sitzungsber. Preuss. Akad. für Wiss. 3 (1910); A.J. Bosch, Am. Math. Mont. **93**, 462 (1986).
- [11] R.A. Horn, C.R. Johnson, *Matrix Analysis*, Cambridge University Press (1985).
- [12] A.J. Laub, K. Meyer, Celestial Mechanics **9**, 213 (1974); V.I. Arnold, *Mathematical Methods of Classical Mechanics*, Springer (1978).
- [13] K. Hammerer, E.S. Polzik, J.I. Cirac, Phys. Rev. A **72**, 052313 (2005).
- [14] A. Botero, B. Reznik, Phys. Rev. A, **67**, 052311 (2002).
- [15] M. Navascues, J. Bae, J.I. Cirac, M. Lewenstein, A. Sanpera, A. Acin, Phys. Rev. Lett. **94**, 010502 (2005).
- [16] G. Giedke, J. Eisert, J.I. Cirac, M.B. Plenio, Quant. Inf. Comp. **3**, 211 (2003).
- [17] J. Fiurasek, L. Mista, Phys. Rev. A **75**, 060302(R) (2007).
- [18] G. Adesso, Phys. Rev. Lett. **97**, 130502 (2006).
- [19] A.S. Holevo, R.F. Werner, Phys. Rev. A, **63**, 032312 (2001).
- [20] J. Eisert, M.M. Wolf, in *Quantum Information with Continuous Variables*, eds. G. Leuchs, N. Cerf, and E. Polzik, Imperial College Press; arXiv:quant-ph/0505151 (2005).
- [21] B. Julsgaard, J. Sherson, J.I. Cirac, Jaromir Fiurasek, E.S. Polzik, Nature **432**, 482 (2004).
- [22] A. Serafini, J. Eisert, M.M. Wolf, Phys. Rev. A **71**, 012320 (2005); M.M. Wolf, D. Perez-Garcia, G. Giedke, Phys. Rev. Lett. **98**, 130501 (2007); F. Caruso, V. Giovannetti, A.S. Holevo, New Journal of Physics **8**, 310 (2006).
- [23] A.S. Holevo, Prob. Inf. Trans. **43**, 1 (2007); arXiv:quant-ph/0607051.